

# THE DUAL SPACE OF PRECOMPACT GROUPS

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**ABSTRACT.** For any topological group  $G$  the dual object  $\widehat{G}$  is defined as the set of equivalence classes of irreducible unitary representations of  $G$  equipped with the Fell topology. If  $G$  is compact,  $\widehat{G}$  is discrete. In an earlier paper we proved that  $\widehat{G}$  is discrete for every metrizable precompact group, i.e. a dense subgroup of a compact metrizable group. We generalize this result to the case when  $G$  is an almost metrizable precompact group.

## 1. INTRODUCTION

For a topological group  $G$  let  $\widehat{G}$  be the set of equivalence classes of irreducible unitary representations of  $G$ . The set  $\widehat{G}$  can be equipped with a natural topology, the so-called Fell topology (see Section 2 for a definition).

A topological group  $G$  is *precompact* if it is isomorphic (as a topological group) to a subgroup of a compact group  $H$  (we may assume that  $G$  is dense in  $H$ ). If  $H$  is compact, then  $\widehat{H}$  is discrete. If  $G$  is a dense subgroup of  $H$ , the natural mapping  $\widehat{H} \rightarrow \widehat{G}$  is a bijection but in general need not be a homeomorphism. Moreover, for every countable non-metrizable precompact group  $G$  the space  $\widehat{G}$  is not discrete [12, Theorem 5.1], and every non-metrizable compact group  $H$  has a dense subgroup  $G$  such that  $\widehat{G}$  is not discrete [12, Theorem 5.2]. (The Abelian case was considered in

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[5, 6, 14]). On the other hand, if  $G$  is a precompact metrizable group, then  $\widehat{G}$  is discrete [12, Theorem 4.1]. (The Abelian case was considered in [2, 4]). The aim of the present paper is to generalize this result to the almost metrizable case:  $\widehat{G}$  is discrete for every almost metrizable precompact topological group  $G$ . A topological group  $G$  is *almost metrizable* if it has a compact subgroup  $K$  such that the quotient space  $G/K$  is metrizable. According to Pasynkov's theorem [1, 4.3.20], a topological group is almost metrizable if and only if it is feathered in the sense of Arhangel'skii.

We reduce the almost metrizable case to the metrizable case considered in [12, Theorem 4.1].

## 2. PRELIMINARIES: FELL TOPOLOGIES

All topological spaces and groups that we consider are assumed to be Hausdorff. For a (complex) Hilbert space  $\mathcal{H}$  the unitary group  $U(\mathcal{H})$  of all linear isometries of  $\mathcal{H}$  is equipped with the strong operator topology (this is the topology of pointwise convergence). With this topology,  $U(\mathcal{H})$  is a topological group.

A *unitary representation*  $\rho$  of the topological group  $G$  is a continuous homomorphism  $G \rightarrow U(\mathcal{H})$ , where  $\mathcal{H}$  is a complex Hilbert space. A closed linear subspace  $E \subseteq \mathcal{H}$  is an *invariant* subspace for  $\mathcal{S} \subseteq U(\mathcal{H})$  if  $ME \subseteq E$  for all  $M \in \mathcal{S}$ . If there is a closed subspace  $E$  with  $\{0\} \subsetneq E \subsetneq \mathcal{H}$  which is invariant for  $\mathcal{S}$ , then  $\mathcal{S}$  is called *reducible*; otherwise  $\mathcal{S}$  is *irreducible*. An *irreducible representation* of  $G$  is a unitary representation  $\rho$  such that  $\rho(G)$  is irreducible.

If  $\mathcal{H} = \mathbb{C}^n$ , we identify  $U(\mathcal{H})$  with the *unitary group of order  $n$* , that is, the compact Lie group of all complex  $n \times n$  matrices  $M$  for which  $M^{-1} = M^*$ . We denote this group by  $\mathbb{U}(n)$ .

Two unitary representations  $\rho : G \rightarrow U(\mathcal{H}_1)$  and  $\psi : G \rightarrow U(\mathcal{H}_2)$  are *equivalent* if there exists a Hilbert space isomorphism  $M : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $\rho(x) = M^{-1}\psi(x)M$  for all  $x \in G$ . The *dual object* of a topological group  $G$  is the set  $\widehat{G}$  of equivalence classes of irreducible unitary representations of  $G$ .

If  $G$  is a precompact group, the Peter-Weyl Theorem (see [15]) implies that all irreducible unitary representation of  $G$  are finite-dimensional and determine an embedding of  $G$  into the product of unitary groups  $\mathbb{U}(n)$ .

If  $\rho : G \rightarrow U(\mathcal{H})$  is a unitary representation, a complex-valued function  $f$  on  $G$  is called a *function of positive type* (or *positive-definite function*) *associated with*  $\rho$  if there exists a vector  $v \in \mathcal{H}$  such that  $f(g) = (\rho(g)v, v)$  (here  $(\cdot, \cdot)$  denotes the inner product in  $\mathcal{H}$ ). We denote by  $P'_\rho$  the set of all functions of positive type associated with  $\rho$ . Let  $P_\rho$  be the convex cone generated by  $P'_\rho$ , that is, the set of sums of elements of  $P'_\rho$ .

Let  $G$  be a topological group,  $\mathcal{R}$  a set of equivalence classes of unitary representations of  $G$ . The *Fell topology* on  $\mathcal{R}$  is defined as follows: a typical neighborhood of  $[\rho] \in \mathcal{R}$  has the form

$$W(f_1, \dots, f_n, C, \epsilon) = \{[\sigma] \in \mathcal{R} : \exists g_1, \dots, g_n \in P_\sigma \forall x \in C |f_i(x) - g_i(x)| < \epsilon\},$$

where  $f_1, \dots, f_n \in P_\rho$  (or  $\in P'_\rho$ ),  $C$  is a compact subspace of  $G$ , and  $\epsilon > 0$ . In particular, the Fell topology is defined on the dual object  $\widehat{G}$ . If  $G$  is locally compact, the Fell topology on  $\widehat{G}$  can be derived from the Jacobson topology on the primitive ideal space of  $C^*(G)$ , the  $C^*$ -algebra of  $G$  [7, section 18], [3, Remark F.4.5].

Every onto homomorphism  $f : G \rightarrow H$  of topological groups gives rise to a continuous injective dual map  $\hat{f} : \widehat{H} \rightarrow \widehat{G}$ . A mapping  $h : X \rightarrow Y$  between topological

spaces is *compact-covering* if for every compact set  $L \subset Y$  there exists a compact set  $K \subset X$  such that  $h(K) = L$ .

**Lemma 2.1.** *If  $f : G \rightarrow H$  is a compact-covering onto homomorphism of topological groups, the dual map  $\hat{f} : \hat{H} \rightarrow \hat{G}$  is a homeomorphic embedding.*

*Proof.* This easily follows from the definition of Fell topology.  $\square$

Let  $\pi$  be a unitary representation of a topological group  $G$  on a Hilbert space  $\mathcal{H}$ . Let  $F \subseteq G$  and  $\epsilon > 0$ . A unit vector  $v \in \mathcal{H}$  is called  $(F, \epsilon)$ -invariant if  $\|\pi(g)v - v\| < \epsilon$  for every  $g \in F$ .

A topological group  $G$  has *property (T)* if and only if there exists a pair  $(Q, \epsilon)$  (called a *Kazhdan pair*), where  $Q$  is a compact subset of  $G$  and  $\epsilon > 0$ , such that for every unitary representation  $\rho$  having a unit  $(Q, \epsilon)$ -invariant vector there exists a non-zero invariant vector. Equivalently,  $G$  has property (T) if and only if the trivial representation  $1_G$  is isolated in  $\mathcal{R} \cup \{1_G\}$  for every set  $\mathcal{R}$  of equivalence classes of unitary representations of  $G$  without non-zero invariant vectors [3, Proposition 1.2.3].

Compact groups have property (T) [3, Proposition 1.1.5], but countable Abelian precompact groups do not have property (T) [12, Theorem 6.1].

We refer to Fell's papers [9, 10], the classical text by Dixmier [7] and the recent monographs by de la Harpe and Valette [13], and Bekka, de la Harpe and Valette [3] for basic definitions and results concerning Fell topologies and property (T).

### 3. ALMOST METRIZABLE GROUPS

If  $A$  is a subset of a topological space  $X$ , the *character*  $\chi(A, X)$  of  $A$  in  $X$  is the least cardinality of a base of neighborhoods of  $A$  in  $X$ . (If this definition leads to a finite value of  $\chi(A, X)$ , we replace it by  $\omega$ , the first infinite cardinal, and similarly for

other cardinal invariants.) If  $A$  is a closed subset of a compact space  $X$ , the character  $\chi(A, X)$  equals the *pseudocharacter*  $\psi(A, X)$  – the least cardinality of a family  $\gamma$  of open subsets of  $X$  such that  $\bigcap \gamma = A$ . In particular, if  $A$  is a closed  $G_\delta$ -subset of a compact space  $X$ , then  $\chi(A, X) = \omega$ .

If  $K$  is a compact subgroup of a topological group, then  $G/K$  is metrizable if and only if  $\chi(K, G) = \omega$  [1, Lemma 4.3.19]. Let  $G$  be an almost metrizable topological group,  $\mathcal{K}$  the collection of all compact subgroups  $K \subset G$  such that  $\chi(K, G) = \omega$ . Then for every neighborhood  $O$  of the neutral element there is  $K \in \mathcal{K}$  such that  $K \subset O$  [1, Proposition 4.3.11]. We now show that if  $G$  is additionally  $\omega$ -narrow, then  $K$  can be chosen normal (in the algebraic sense). Recall that a topological group  $G$  is  $\omega$ -narrow [1] if for every neighborhood  $U$  of the neutral element there exists a countable set  $A \subset G$  such that  $AU = G$ .

**Lemma 3.1.** *Let  $G$  be an  $\omega$ -narrow almost metrizable group,  $\mathcal{N}$  the collection of all normal (= invariant under inner automorphisms) compact subgroups  $K$  of  $G$  such that the quotient group  $G/K$  is metrizable (equivalently,  $\chi(K, G) = \omega$ ). Then for every neighborhood  $O$  of the neutral element there exists  $K \in \mathcal{N}$  such that  $K \subset O$ .*

*Proof.* Let  $L \subset O$  be a compact subgroup of  $G$  such that the quotient space  $G/L = \{xL : x \in G\}$  is metrizable. It suffices to prove that  $K = \bigcap \{gLg^{-1} : g \in G\}$ , the largest normal subgroup of  $G$  contained in  $L$ , belongs to  $\mathcal{N}$ .

There exists a compatible metric on  $G/L$  which is invariant under the action of  $G$  by left translations. To construct such a metric, consider a countable base  $U_1, U_2, \dots$  of neighborhoods of  $L$  in  $G$ . We may assume that for each  $n$  we have  $U_n = U_n^{-1} = U_n L$  and  $U_{n+1}^2 \subset U_n$ . Let  $\gamma_n = \{gU_n : g \in G\}$ . The open cover  $\gamma_n$  of  $G$  is invariant under left  $G$ -translations and under right  $L$ -translations, and  $\gamma_{n+1}$  is a barycentric refinement

of  $\gamma_n$ . The pseudometric on  $G$  that can be constructed in a canonical way from the sequence  $(\gamma_n)$  of open covers (see [8, Theorem 8.1.10]) gives rise to a compatible  $G$ -invariant metric on  $G/L$ . A similar construction was used in [1, Lemma 4.3.19].

If an  $\omega$ -narrow group transitively acts on a metric space  $X$  by isometries, then  $X$  is separable [1, 10.3.2]. Thus  $X = G/L$  is separable. Let  $Y$  be a dense countable subset of  $X$ . Then  $K = \{g \in G : gx = x \text{ for every } x \in X\} = \{g \in G : gx = x \text{ for every } x \in Y\}$  is a  $G_\delta$ -subset of  $L$ , hence  $\chi(K, L) = \omega$ . It follows that  $\chi(K, G) \leq \chi(K, L)\chi(L, G) = \omega$  ([8, Exercise 3.1.E]).  $\square$

#### 4. MAIN THEOREM

**Theorem 4.1.** *If  $G$  is a precompact almost metrizable group, then  $\widehat{G}$  is discrete.*

*Proof.* Let  $\rho$  be an irreducible unitary representation of  $G$ . We must prove that  $[\rho]$  is isolated in  $\widehat{G}$ . It suffices to find a discrete open subset  $D \subset \widehat{G}$  such that  $[\rho] \in D$ .

Precompact groups are  $\omega$ -narrow, so Proposition 3.1 applies to  $G$ . Let  $\mathcal{N}$ , as above, be the collection of all normal compact subgroups  $K \subset G$  such that  $\chi(K, G) = \omega$ . Then  $\mathcal{N}$  is closed under countable intersections, and it follows from Proposition 3.1 that for every  $G_\delta$ -subset  $A$  of  $G$  containing the neutral element there exists  $K \in \mathcal{N}$  such that  $K \subset A$ . In particular, there exists  $K \in \mathcal{N}$  such that  $K$  lies in the kernel of  $\rho$ . Let  $D \subset \widehat{G}$  be the set of all classes  $[\sigma] \in \widehat{G}$  such that  $K$  is contained in the kernel of  $\sigma$ . Then  $[\rho] \in D$ . It suffices to verify that  $D$  is open and discrete.

Step 1. We verify that  $D$  is open. Let  $\mathcal{R}$  be the set of equivalence classes of all finite-dimensional unitary representations (which may be reducible) of  $K$  without non-zero invariant vectors. Let  $\tau_n$  be the trivial  $n$ -dimensional representation  $1_K \oplus \cdots \oplus 1_K$  ( $n$  summands) of  $K$ ,  $n = 1, 2, \dots$ . In the notation of section 2,  $P_{\tau_n}$  does not depend

on  $n$  and is the set of non-negative constant functions on  $K$ . It follows that in the space  $\mathcal{S} = \mathcal{R} \cup \{[\tau_n] : n = 1, 2, \dots\}$ , equipped with the Fell topology, the points  $[\tau_n]$  are indistinguishable: any open set containing one of these points contains all the others. Since  $K$  has property (T),  $[\tau_1] = [1_K]$  is not in the closure of  $\mathcal{R}$ . Therefore  $\mathcal{R}$  is closed in  $\mathcal{S}$  and  $\mathcal{S} \setminus \mathcal{R}$  is open in  $\mathcal{S}$ .

We claim that for every irreducible unitary representation  $\sigma$  of  $G$  the class of the restriction  $\sigma|_K$  belongs to  $\mathcal{S}$ . In other words, the claim is that  $\sigma|_K$  is trivial if it admits a non-zero invariant vector. Let  $V$  be the (finite-dimensional) space of the representation  $\sigma$ . For  $g \in G$  and  $x \in V$  we write  $gx$  instead of  $\sigma(g)x$ . The space  $V' = \{x \in V : gx = x \text{ for all } g \in K\}$  of all  $K$ -invariant vectors is  $G$ -invariant. Indeed, if  $x \in V'$ ,  $g \in G$  and  $h \in K$ , then  $g^{-1}hgx = x$  because  $g^{-1}hg \in K$  and  $x$  is  $K$ -invariant. It follows that  $hgx = gx$  which proves that  $gx \in V'$ . Since  $\sigma$  is irreducible, either  $V' = \{0\}$  or  $V' = V$ . Accordingly, either  $\sigma|_K$  admits no non-zero invariant vectors or else is trivial.

We have just proved that the restriction map  $r : \widehat{G} \rightarrow \mathcal{S}$  is well-defined. Clearly  $r$  is continuous, and therefore  $D = r^{-1}(\mathcal{S} \setminus \mathcal{R})$  is open in  $\widehat{G}$ .

Step 2. We verify that  $D$  is discrete. Let  $p : G \rightarrow G/K$  be the quotient map. Then  $D$  is the image of the dual map  $\hat{p} : \widehat{G/K} \rightarrow \widehat{G}$ . According to [12, Theorem 4.1], the dual space of a metrizable precompact group is discrete. Thus  $\widehat{G/K}$  is discrete. Since  $p$  is a perfect map, it is compact-covering, and Lemma 2.1 implies that  $\hat{p} : \widehat{G/K} \rightarrow \widehat{G}$  is a homeomorphic embedding. Therefore,  $D = \hat{p}(\widehat{G/K})$  is discrete.  $\square$

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